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Semi-Analytical Method for 1D Solidification Stresses

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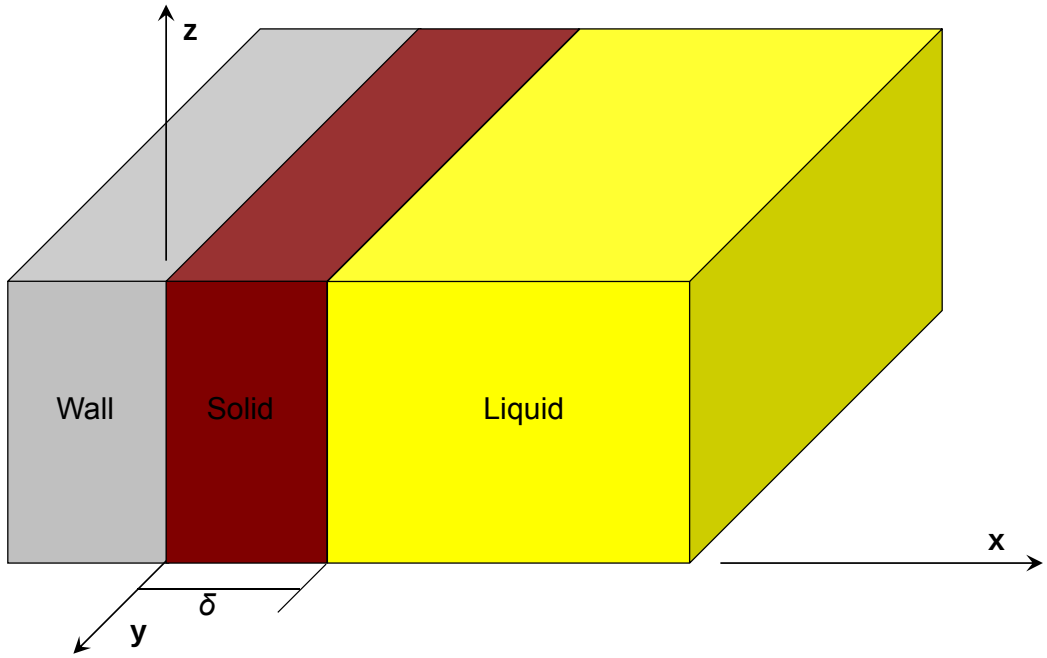
Objectives

- Develop method for computing mechanical response of infinite solidifying plate
- Generalization of elastoplastic Stefan problem (Weiner & Boley, JMPS 1963)
  - Arbitrary thermal history
  - Arbitrary mechanical behavior
- Implement into CON1D

Domain: Semi-Infinite Solidifying Plate



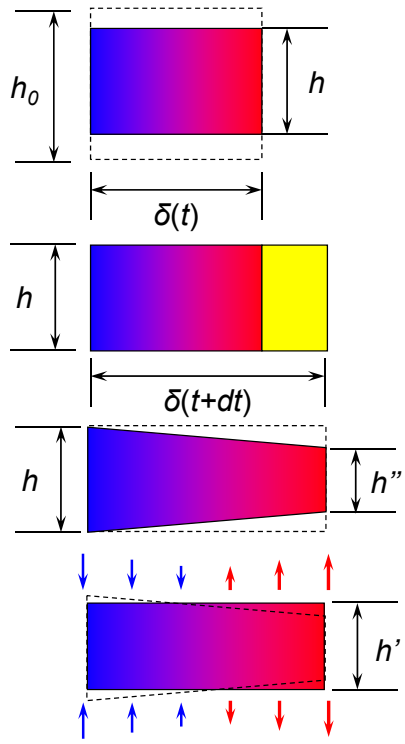
Boundary Conditions

- Consider any straight-line slice of material
- Neighboring slices constrain it to remain a straight line, but dimensions may change
  - Physically: zero resultant force on cross section
  - Mathematically: $\int_0^\delta \sigma dx = 0$
- This includes preventing bending
  - Physically: zero resultant moment on cross section
  - Mathematically: $\int_0^\delta \sigma x dx = 0$
- Stress tensor has the form

$$[\sigma] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

where $\sigma = \sigma(x,t)$ is the hoop stress profile

The Key to Thermal Stress Problems is Understanding Constraint



1. Surface layer solidifies and shrinks and is in equilibrium

2. An instant later, a new layer of material solidifies at same thickness as surface layer

3a. As it cools, the new layer tries to shrink, but the surface no longer wants to shrink

3b. The constraint to a straight line fights the subsurface shrinkage, causing surface compression and subsurface tension. Since the domain is infinite, the liquid is pushed to infinity

Stress

- Stress tensor is $[\sigma]=\begin{bmatrix}0 & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & \sigma\end{bmatrix}$ $\qquad \sigma=\sigma(x,t)$
- Deviatoric stress $[\sigma']=[\sigma]-p[\boldsymbol{I}]=\frac{\sigma}{3}\begin{bmatrix}-2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$
- Pressure stress $p=\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})=\frac{2}{3}\sigma$
- von Mises stress $\bar{\sigma}=\sqrt{\frac{3}{2}\boldsymbol{\sigma}':\boldsymbol{\sigma}'}=|\sigma|$
- Equilibrium is satisfied without body forces

$$\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{b}=0$$

Total Strain

- Constraint gives strain tensor of

$$[\varepsilon]=\begin{bmatrix}\varepsilon_{\parallel} & 0 & 0\\ 0 & \varepsilon_{\perp} & 0\\ 0 & 0 & \varepsilon_{\perp}\end{bmatrix}$$

$\varepsilon_{\perp}=\varepsilon_{\perp}(x,t)$ Strain perpendicular to solidification direction

$\varepsilon_{\parallel}=\varepsilon_{\parallel}(x,t)$ Strain parallel to solidification direction

- Barré de Saint-Venant compatibility

$$\nabla\times\varepsilon\times\nabla=0$$

  – 6 equations, all trivially satisfied except

$$\frac{\partial^2\varepsilon_{yy}}{\partial x^2}=\frac{\partial^2\varepsilon_{zz}}{\partial x^2}=\frac{\partial^2\varepsilon_{\perp}}{\partial x^2}=0$$

  – Integrate twice to get $\varepsilon_{\perp}(x,t)=F(t)+xG(t)$

Total Strain

- Geometrically, bending needs

$$\varepsilon_{\perp}(x,t)=xA+x^2B+\cdots$$

- Take G(t) = 0 to prevent bending
- Total perpendicular strain is a constant over the domain that changes with time

$$\varepsilon_{\perp}=\varepsilon_{\perp}(t)$$

- Additive decomposition of strain rates

$$\dot{\varepsilon}=\dot{\varepsilon}^{th}+\dot{\varepsilon}^{el}+\dot{\varepsilon}^{ie}$$

Thermoelasticity

- Elasticity $\dot{\boldsymbol{\varepsilon}}^{el} = \frac{1+\nu}{E}\dot{\boldsymbol{\sigma}} - \frac{\nu}{E}\mathrm{tr}(\dot{\boldsymbol{\sigma}})\boldsymbol{I}$

E Young's modulus

ν Poisson's ratio

$$\left[\dot{\boldsymbol{\varepsilon}}^{el}\right] = \frac{\dot{\sigma}}{E}\begin{bmatrix} -2\nu & 0 & 0 \\ 0 & 1-\nu & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}$$

- Thermal expansion

$$\dot{\boldsymbol{\varepsilon}}^{th} = \alpha\dot{T}\boldsymbol{I}$$

$\dot{T}$ Time derivative of temperature

α Coefficient of thermal expansion

$$\left[\dot{\boldsymbol{\varepsilon}}^{th}\right] = \alpha\dot{T}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inelasticity

- General case: inelastic strain rates of

$$\left[\dot{\boldsymbol{\varepsilon}}^{ie}\right] = \begin{bmatrix} \dot{\varepsilon}_{\parallel}^{ie} & 0 & 0 \\ 0 & \dot{\varepsilon}_{\perp}^{ie} & 0 \\ 0 & 0 & \dot{\varepsilon}_{\perp}^{ie} \end{bmatrix} \qquad \dot{\varepsilon}_{\parallel}^{ie} + 2\dot{\varepsilon}_{\perp}^{ie} = 0$$

- Assume material obeys von Mises yielding with isotropic hardening
  - Prandtl-Reuss equations

$$\dot{\boldsymbol{\varepsilon}}^{ie} = \frac{3}{2}\dot{\bar{\varepsilon}}^{ie}\frac{\boldsymbol{\sigma}'}{\bar{\sigma}} \qquad \left[\dot{\boldsymbol{\varepsilon}}^{ie}\right] = \dot{\bar{\varepsilon}}^{ie}\,\mathrm{sign}(\sigma)\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$\dot{\bar{\varepsilon}}^{ie}(\boldsymbol{\sigma}', T, \ldots) \geq 0$ Effective inelastic strain rate

$\mathrm{sign}(\sigma) = \sigma/|\sigma|$ Sign of stress

Material Response

- A pair of differential equations characterizes the behavior of the solidifying plate

$$\dot{\varepsilon}_{\parallel} = -2\frac{\nu}{E}\dot{\sigma} + \alpha\dot{T} - \dot{\bar{\varepsilon}}^{ie}\,\mathrm{sign}(\sigma) \qquad \textbf{(1)}$$

$$\dot{\varepsilon}_{\perp} = \frac{1}{B}\dot{\sigma} + \alpha\dot{T} + \frac{1}{2}\dot{\bar{\varepsilon}}^{ie}\,\mathrm{sign}(\sigma) \qquad \textbf{(2)}$$

$B = E/(1-\nu)$

Biaxial modulus

- Mostly interested in "perpendicular" equation
  - Shrinkage profiles
  - Longitudinal cracking

Constraint

- Constraint equations:

$$\int_0^{\delta} \sigma dx = 0 \qquad \int_0^{\delta} \sigma x dx = 0$$

- Rate forms (valid because newly solidified material is always stress-free):

$$\int_0^{\delta} \dot{\sigma} dx = 0 \qquad \int_0^{\delta} \dot{\sigma} x dx = 0$$

- Constraint against bending is enforced with compatibility condition
- Integrate equation (2) and use $\int_0^{\delta} \dot{\sigma} dx = 0$

Constraint

- Integral of equation (2):

$$\int_0^\delta \dot{\varepsilon}_\perp B\,\mathrm{d}x = \cancel{\int_0^\delta \dot{\sigma}\,\mathrm{d}x} + \int_0^\delta B\alpha\dot{T}\,\mathrm{d}x + \int_0^\delta B\frac{\dot{\bar{\varepsilon}}^{ie}}{2}\mathrm{sign}(\sigma)\,\mathrm{d}x$$

- Total strain rate is uniform, so:

$$\dot{\varepsilon}_\perp \int_0^\delta B\,\mathrm{d}x = \int_0^\delta B\alpha\dot{T}\,\mathrm{d}x + \int_0^\delta B\frac{\dot{\bar{\varepsilon}}^{ie}}{2}\mathrm{sign}(\sigma)\,\mathrm{d}x$$

  – Explicit relation for total strain rate

- If elastic properties are uniform,

$$\dot{\varepsilon}_\perp = \frac{1}{\delta}\int_0^\delta \alpha\dot{T}\,\mathrm{d}x + \frac{1}{\delta}\int_0^\delta \frac{\dot{\bar{\varepsilon}}^{ie}}{2}\mathrm{sign}(\sigma)\,\mathrm{d}x$$

  – Total strain rate is the sum of the spatial averages of the thermal and inelastic strain rates

Explicit Algorithm

- Start
  – Given temperature, stress, strains
  – Find new thermal strain rates, spatially integrate
  – Find new inelastic strain rates, spatially integrate
  – Total strain rate $\dot{\varepsilon}_\perp \int_0^\delta B\,\mathrm{d}x = \int_0^\delta B\alpha\dot{T}\,\mathrm{d}x + \int_0^\delta B\frac{\dot{\bar{\varepsilon}}^{ie}}{2}\mathrm{sign}(\sigma)\,\mathrm{d}x$
  – Stress rate $\dot{\varepsilon}_\perp = \frac{1}{B}\dot{\sigma} + \alpha\dot{T} + \frac{1}{2}\dot{\bar{\varepsilon}}^{ie}\,\mathrm{sign}(\sigma)$
  – Advance all fields with calculated rates

$$\psi(x,t+\Delta t) = \psi(x,t) + \Delta t\,\dot{\psi}(x,t)$$

  – New time step

Practical Extension

- Assume Newtonian fluid

μ Viscosity

$$\dot{\boldsymbol{\varepsilon}}^{\ell} = \frac{1}{2\mu}\boldsymbol{\sigma}' \qquad \left[\dot{\boldsymbol{\varepsilon}}^{\ell}\right] = \frac{\sigma}{6\mu}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Strain decomposition, f = fraction liquid

$$\dot{\boldsymbol{\varepsilon}} = (1-f)\left(\dot{\boldsymbol{\varepsilon}}^{th} + \dot{\boldsymbol{\varepsilon}}^{el} + \dot{\boldsymbol{\varepsilon}}^{ie}\right) + f\dot{\boldsymbol{\varepsilon}}^{\ell}$$

- Same analysis; total strain rate is

$$\dot{\varepsilon}_{\perp}\int_0^{\delta} B\,\mathrm{d}x = \int_0^{\delta}(1-f)B\alpha\dot{T}\,\mathrm{d}x + \int_0^{\delta}(1-f)B\frac{\dot{\bar{\varepsilon}}^{ie}}{2}\mathrm{sign}(\sigma)\,\mathrm{d}x + \int_0^{\delta} fB\frac{\sigma}{6\mu}\mathrm{d}x$$

- Now this can be used to evaluate hot tearing

Example: Weiner-Boley Problem

$$T_s(x,t) = T_w + \frac{T_m - T_w}{\mathrm{erf}(\phi)}\mathrm{erf}\left(\frac{x}{\sqrt{4\alpha_D t}}\right)$$

Temperature in solid

$$\delta(t) = \phi\sqrt{4\alpha_D t}$$

Shell thickness

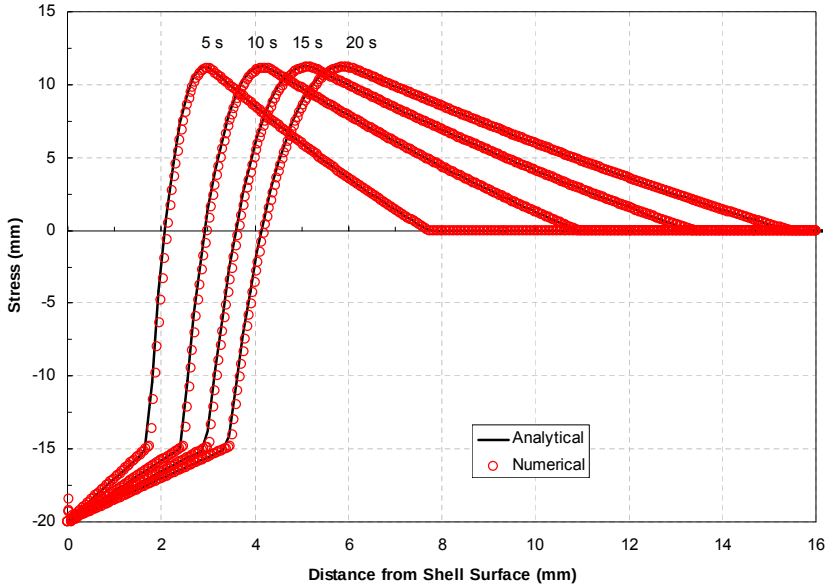
$$\sigma_Y(T) = \sigma_{Yw}\frac{T_m - T}{T_m - T_w}$$

Yield stress

$$\dot{\bar{\varepsilon}}^{ie} = \begin{cases} A\left(|\sigma| - \sigma_Y(T)\right) & \text{if } |\sigma| > \sigma_Y(T) \\ 0 & \text{otherwise} \end{cases}$$

Constitutive law

Example: Weiner-Boley Problem



$\Delta x = 0.04$ mm $\Delta t = 0.1$ ms

Conclusions

- Weiner and Boley problem has been generalized for arbitrary thermal and mechanical behavior
- Currently implemented as MATLAB script
  - Can match analytical solution
- Look for implementation in CON1D in future
  - Better shrinkage profiles!

Acknowledgements

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